FINAL: ALGEBRAIC NUMBER THEORY

The total points is **100**.

A **ring** would mean a **commutative ring with identity**.

(1) (10+10+10=30 points) Prove or disprove
   (a) Let R be a Dedekind domain with fraction field K, L/K a finite separable extension and R' be the integral closure of R in L. Then every subring of R' containing R is a Dedekind domain.
   (b) Let $\mathcal{L}$ be a lattice in a finite dimensional vector space V. Then $\mathcal{L}$ is free abelian group of finite rank.
   (c) Let R be a Dedekind domain. If the class group of R_P for every maximal ideal P of R is trivial then class group of R is trivial.

(2) (20 points) Let R be a normal domain and an integral domain S be an integral extension of R. Let K and L be fraction fields of R and S respectively such that $[L : K] < \infty$. Show that $T_{L/K}(S) \subset R$ and $N_{L/K}(S) \subset R$.

(3) (15 points) Let $L = \mathbb{Q}(\sqrt{5}, \sqrt{7}, \sqrt[3]{11})$ and R the ring of integers of L, compute all the primes of $\mathbb{Z}$ ramified in R.

(4) (15 points) Let K be a number field and R be its ring of integers. Let I be an ideal of R. Show that if the norm of the ideal $N(I)$ is a prime ideal of $\mathbb{Z}$ then I is a prime ideal of R.

(5) (20 points) Let p be an odd prime and $[K : \mathbb{Q}] = p$. Let U be the group of units in the ring of integers of K. Show that the torsion subgroup of U has order 2 and rank of U is at least $(p-1)/2$.